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Using B-spline for shape preserving linear singularly perturbed boundary value problems

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ABSTRACT

This paper considers the B-spline interpolation method for solving shape-preserving linear singularly perturbed boundary value problem of the form: $\varepsilon y'' + p(x)y' + q(x)y = r(x),\ y(a) = \alpha,\ y(b) = \beta.\quad where\ 0 < \varepsilon \le 1$

A new method computationally is used to find the best derivative at the boundary points to apply B-spline. Finally, the results are compared to prevously published example.

Keywords: Shape-preserving; B-spline; Linear Singularly; Perturbed.

1. Introduction

The general form of singularly perturbed boundary value problem has the form

$$\varepsilon y'' + p(x)y' + q(x)y = r(x),\quad y(a) = \alpha,\ y(b) = \beta,\ and\ 0 < \varepsilon \le 1$$

, Such that $p(x), q(x)$ *, and* $r(x)$ are smooth and bounded functions. This problem occurs in many engineering fields, especially those interested in applied mathematics such as quantum mechanics, chemical reactor theory, optimal control, and reaction-diffusion process [1,2].

First of all, the meaning of perturbed theory comprises the mathematical methods used to find an approximate solution to a problem that cannot be solved precisely. Perturbation theory is applicable if the problem can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem [3].

Second, the shape-preserving means that the interpolated polynomial has the same property as the original function, for example, smoothness, increasing or decreasing behaviour. Many people think that if the first derivative terms are not given at the boundary value problems, then the B-spline interpolation is not used to solve the singularly perturbed boundary value problems. This paper utilizes a scheme that approximates the first derivative at the boundary value problem using the second-degree polynomial that passes the nodes $(x_i\,,\ f(x_i))$ [4].

In brief, if we have S as a polynomial spline of the second degree, and we have a free parameter $d = S'(a)$ *, where* $\Omega_n = \{x_0, x_1, x_2, ..., x_n\}$ is a set of nodes of a partition on the interval [a, b], such that $a = x_0 < x_1 < x_2 < .... < x_n = b$, and $f(x_i) = y_i$ is given for all $i = 0, 1, 2, 3, ..., n$, and suppose that $S_2(x)$ is continuous on [a, b] and differentiable and that means $S_2 \in C^1$[a, b], then we can find $S'(a) = S'(x_0) = d$ such that $S \in S_2(\Omega_n,\ f)$ is shape-preserving.

For more details, denote S_i is considered a second-degree polynomial on the interval $I_i = [x_i,\ x_{i+1}]$ as a restriction of S as that $S_i = S\big|_{I_i}$, also we assume $1,\ (x-x_i),\ (x-x_i)^2$ as a basis, this means that at any nodes x_i $where\ i = 0, 1, 2, \ldots n$ we get:

$$S_i(x) = S(x_i) + S'(x_i)(x-x_i) + \left[\frac{S[x_{i+1}, x_i] - S'(x_i)}{x_{i+1}-x_i}\right](x-x_i)^2$$

Collect and put all of the given information in a matrix forms denoted by q, where $q_1 = 0,\ and\ q_{i+1} = 2\left(\frac{y_{i+1}-y_i}{x_{i+1}-x_i}\right) - q_i$.

Now, a Matlab program can be used to solve this problem. Therefore, we have to decide that whether the function is monotone decreasing or increasing if denoted as an interval which contains the best of the derivative which is:

$$d = \begin{cases} \max\{q_i : i = 2k\} \le d \le -\max\{q_i : i = 2k-1\} \\ \quad where\quad k = 1, 2, \ldots n,\ if\quad S\ \ mon.\ \ dec. \\ -\min_{i\ add}(q_i) \le d \le \min_{i\ even}(q_i) \\ \quad if\quad S\ \ mon.\ \ inc. \end{cases}$$

2. Example

Let $f = \frac{1}{x}$, $and\ \ x = [1:160]$, then by the previous program, find the interval which contains the best derivatives $d = [-.7726,\ -7725]$. More information is presented in previous work [4].

In the same way, we can find $S'(x_n) = S'(b)$. Then, the B-spline can be applied to approximate the solution of the "Shape Preserving Linear Singularly Perturbed Boundary Value Problems."

3. Method Description

First, the problem will be solved by the cubic B-spline method to find the best approximation nodes $(x_i\ ,\ f(x_i))$ using the previous section. The general form of the linear singularly perturbed boundary value problem [5] as the follow form:

$$\varepsilon y'' + p(x)y' + q(x)y = r(x),$$
$$where\ \ y(a) = \alpha,\ \ y(b) = \beta,\ \ and\ \ 0 < \varepsilon \le 1$$

The cubic B-spline basis will be used to solve this problem as the follow form:

$$B_{3,j}(x) = \frac{1}{6h^3}\begin{cases} (x-x_i)^3 \\ \quad where\ \ x \in [x_i,\ x_{i+1}] \\ h^3 + 3h^2(x-x_i) + 3h(x-x_i)^2 - 3(x-x_i)^3 \\ \quad where\ \ x \in [x_{i+1},\ x_{i+2}] \\ h^3 + 3h^2(x_{i+3}-x) + 3h(x_{i+3}-x)^2 - 3(x_{i+3}-x)^3 \\ \quad where\ \ x \in [x_{i+2},\ x_{i+3}] \\ (x_{i+4}-x)^3 \\ \quad where\ \ x \in [x_i,\ x_{i+1}] \end{cases}$$

Then construct the following system using the previous basis in the general form of the linear singularly perturbed boundary value problem.

$$\begin{bmatrix} 1/6 & 2/3 & 1/6 & 0 & 0 & . & . & . & . & 0 \\ u_0 & v_0 & s_0 & 0 & 0 & . & . & . & . & 0 \\ 0 & u_1 & v_1 & s_1 & 0 & 0 & . & . & . & . \\ 0 & 0 & u_2 & v_2 & s_2 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & 0 & u_i & v_i & s_i & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 & u_n & v_n & s_n \\ 0 & 0 & . & . & . & . & 0 & 1/6 & 2/3 & 1/6 \end{bmatrix} \begin{bmatrix} C_{-3} \\ C_{-2} \\ C_{-1} \\ C_0 \\ . \\ . \\ . \\ . \\ . \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} \alpha \\ r(x_0) \\ r(x_1) \\ . \\ . \\ . \\ . \\ . \\ r(x_n) \\ \beta \end{bmatrix}$$

$$where\ u_i = \varepsilon/h^2 - p(x_i)/2h + q(x_i)/6\quad ,$$
$$v_i = -2\varepsilon/h^2 + 2q(x_i)/3\ \ ,$$
$$and\quad s_i = \varepsilon/h^2 + p(x_i)/2h + q(x_i)/6\ \ ,$$
$$i = 0, 1, 2, 3, \ldots\ldots.., n$$

Also, the first and the last raws are the boundary conditions $y(a) = \alpha,\ and\ \ y(b) = \beta$.

This system $AC = B$ can be solved to find C, and substitute it in the cubic B-spline interpolation form $S_f(x) = \sum_{i=-3}^{n-1} C_i B_{3,i}(x)$, and then the $f(x_i)$ can be approximated.

Finding the best derivative at the boundary value can be applied and then recognizing it can be applied, which gives a new information for the application of B-spline fifth degree interpolation method. Also, the error of the cubic B-spline interpolation results in the previous section will be compared with the fifth degree B-spline interpolation results in the example later to see how dose it gets the best results [6].Therfore, according to the founding information, the fifth degree B-spline can be applied as follow:

$$
B_{5,j}(x) = \frac{1}{120h^5}\begin{cases}
(x-x_i)^5 \\
where \quad x \in [x_i,\ x_{i+1}] \\
h^5 + 5h^4(x-x_{i+1}) + 10h^3(x-x_{i+1})^2 \\
+10h^2(x-x_{i+1})^3 + 5h(x-x_{i+1})^4 \\
-5(x-x_{i+1})^5 \\
where \quad x \in [x_{i+1},\ x_{i+2}] \\
26h^5 + 50h^4(x-x_{i+2}) + 20h^3(x-x_{i+2})^2 \\
-20h^2(x-x_{i+2})^3 - 20h(x-x_{i+2})^4 \\
+10(x-x_{i+2})^5 \\
where \quad x \in [x_{i+2},\ x_{i+3}] \\
26h^5 + 50h^4(x_{i+4}-x) + 20h^3(x_{i+4}-x)^2 \\
-20h^2(x_{i+4}-x)^3 - 20h(x_{i+4}-x)^4 \\
+10(x_{i+4}-x)^5 \\
where \quad x \in [x_{i+3},\ x_{i+4}] \\
h^5 + 5h^4(x_{i+5}-x) + 10h^3(x_{i+5}-x)^2 \\
+10h^2(x_{i+5}-x)^3 + 5h(x_{i+5}-x)^4 \\
-5(x_{i+5}-x)^5 \\
where \quad x \in [x_{i+4},\ x_{i+5}] \\
(x_{i+6}-x)^5 \\
where \quad x \in [x_{i+5},\ x_{i+6}]
\end{cases}
$$

Finally, construct the following system using the previous basis in the general form of the linear singularly perturbed boundary value problem.

Then the fifth degree B-spline system would be:

$$
\begin{bmatrix}
-5/120\,h & -5/12\,h & 0 & 5/12\,h & 5/120\,h & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1/120 & 26/120 & 66/120 & 26/120 & 1/120 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
u_0 & v_0 & s_0 & w_0 & z_0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & u_1 & v_1 & s_1 & w_1 & z_1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & & & & & & & & & & & & \vdots \\
\cdots & \cdots & 0 & u_i & v_i & s_i & w_i & z_i & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & & & & & & & & & & & & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & u_n & v_n & s_n & w_n & z_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1/120 & 26/120 & 66/120 & 26/120 & 1/120 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & -5/120\,h & -5/12\,h & 0 & 5/12\,h & 5/120\,h
\end{bmatrix}
\begin{bmatrix} C_{-5} \\ C_{-4} \\ C_{-3} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ C_{n-1} \end{bmatrix}
=
\begin{bmatrix} \delta \\ \alpha \\ r(x_0) \\ r(x_1) \\ \vdots \\ \vdots \\ \vdots \\ r(x_n) \\ \beta \\ \gamma \end{bmatrix}
$$

$$
Where \quad u_i = \varepsilon/6h^2 - 5p(x_i)/120\,h + q(x_i)/120,
$$
$$
v_i = \varepsilon/3h^2 - 5p(x_i)/12\,h + 26\,q(x_i)/120.
$$
$$
s_i = -\varepsilon/h^2 + 66\,q(x_i)/120,
$$
$$
w_i = \varepsilon/3h^2 + 5p(x_i)/12\,h + 26\,q(x_i)/120 .
$$
$$
z_i = \varepsilon/6h^2 + 5p(x_i)/120\,h + q(x_i)/120 ,
$$
$$
i = 0,\ 1,\ 2,\ 3, \ldots\ldots\ldots, n.,\ and \quad h = 1/n
$$

Also, the first two rows and the last two rows are the boundary conditions.

These are of the form $A\,C = B$, then compute C and substitute it in the formula:

$S_f(x) = \sum_{n-5}^{n-1} C_i B_{5,i}(x)$. This scheme was demonstrated practically in the following example compared to previous study [7].

4. Example

$$
\varepsilon\, y'' + (1+x^2)y' + 2(1+x)y = \frac{1}{2}\exp\left(-\frac{x}{2}\right)\left[(1+x)(3-x) + \frac{\varepsilon}{2}\right]
$$

$$and\ \ y(0)=0,\ y(1)=\exp\left(-\frac{1}{2}\right)-\exp\left(-\frac{7}{3\,\varepsilon}\right)$$

Whose exact solution:

$$y(x)=\exp\left(-\frac{x}{2}\right)-\exp\left[-\frac{x(x^2+3x+3)}{3\varepsilon}\right]$$

The solution of this example will be applying the previous steps and the results of the maximum absolute error $\max|y(x_i)-y_i|$ at nodes points computed at the deferent values of ε , where h=1/n. Furthermore, the differentiating of the boundary values by using the cubic B-spline depending on the values of ε are found in tables 1 and 2, the maximum error tables: n = 256, n=512 respectively.

Table 1. the differentiating of the boundary values by using the cubic B-spline depending on the values of ε , n =256

$\boldsymbol{\varepsilon}$	Kadalajoo& Patidai's method	Aziz &Khan's method	Rajesh K. Bawa's Method	Proposed method: Cubic B-spline / fifth degree B-spline	Proposed method: $\boldsymbol{\delta = f'(a)}$	Proposed method: $\boldsymbol{\gamma = f'(b)}$
1/8	2.20e-04	2.934416e-05	1.007339e-09	1.66118787e-05 9.32321797e-09	7.499998	-030326887
1/16	1.10e-03	1.367532e-04	1.054689e-08	9.08945084e-05 2.44180388e-08	15.50000	-0.30327385
1/32	5.00e-03	5.116162e-04	1.429318e-07	4.42022286e-04 5.24520294e-08	31.50000	-030327015
1/64	2.30e-02	1.991524e-03	2.134858e-06	1.80759128e-03 1.40487648e-06	63.49999	-0.30327105
1/128	----------	8.007187e-03	3.345245e-05	7.62065557e-03 2.43253474e-05	127.4998	-0.30322423

Table 2. the differentiating of the boundary values by using the cubic B-spline depending on the values of ε , n =512

$\boldsymbol{\varepsilon}$	Kadalajoo& Patidai's method	Aziz &Khan's method	Rajesh K. Bawa's Method	Proposed method: Cubic B-spline / fifth degree B-spline	Proposed method: $\boldsymbol{\delta = f'(a)}$	Proposed method: $\boldsymbol{\gamma = f'(b)}$
1/8	5.60e-05	9.834416e-06	6.295980e-11	4.16573269e-06 4.48480752e-09	7.499998	-030326887
1/16	2.70e-04	3.417675e-05	6.596570e-10	2.27226995e-05 1.21952216e-08	15.50000	-0.30327385
1/32	1.20e-03	1.277339e-04	8.926741e-09	1.05016167e-04 9.71263347e-09	31.50000	-030327015
1/64	5.50e-03	4.952798e-04	1.330488e-07	4.49877279e-04 8.56900241e-08	63.49999	-0.30327105
1/128	2.40e-02	1.959883e-03	2.067402e-06	1.86820104e-03 1.69829183e-06	127.4998	-0.30322423

5. Conclusion

A new technique has been introduced in this study, which is an alternative method to solve the linear singularly perturbed boundary value problems. First, the third-degree b-spline has been used to find the approximation of internal nodes. Second, the new technique has been used to approximate the first derivative at the boundary conditions, which uses the natural quadratic interpolation spline. Then the fifth-degree b-spline has been used to approximate the solution of the linear singularly perturbed boundary value problem. Finally, the numerical result is compared with the numerical example in [5]. In the future, the third-degree natural interpolation spline will be studied to approximate a better result.

6. References

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